Semi-invariant submanifolds of normal complex contact metric manifolds

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Abstract

In this paper, we study on semi-invariant submanifolds of normal complex contact metric manifolds. We give the definition of such submanifolds and we obtain useful relations. Moreover, we give the integrability conditions of distributions.

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1 Introduction

Contact geometry could be divided into two parts: real and complex. The geometry of real contact manifolds have been studied from 1960s, and there are a great number of articles in literature. Although the complex contact geometry is old as real contact geometry, there do not exist researches as real case. On the other hand, results have been obtained in complex contact manifolds are different from real contact manifolds. Also complex contact manifolds have several applications in optimal control of entanglements [1]. The Riemannian geometry of complex contact manifolds has a lot of open problems. One of them is the submanifold theory of complex contact manifolds. In this paper, we aim enter to this notion comprehensively.

The definition of a complex contact manifold was given by Kobayashi [2] as follows: Let M be a complex manifold of odd complex dimension $2m+1$ covered by an open covering $\mathcal{A}=\{\mathcal{U}_i\}$ consisting of coordinate neighborhoods. If there is a holomorphic 1-form ω_i on each $\mathcal{U}_i \in \mathcal{A}$ in such a way that for any $\mathcal{U}_i, \mathcal{U}_j \in \mathcal{A}$ and for a holomorphic function f_{ij} on $\mathcal{U}_i \cap \mathcal{U}_j \neq \varnothing$

$$\omega_i \wedge (d\omega_j)^m \neq 0 \text{ in } \mathcal{U}_i,$$

$$\omega_i = f_{ij}\omega_j,\ \mathcal{U}_i \cap \mathcal{U}_j \neq \varnothing,$$

then the set $\{(\omega_i, \mathcal{U}_i) \mid \mathcal{U}_i \in \mathcal{A}\}$ of local structures is called complex contact structure and with this structure M is called a complex contact manifolds.

In this definition there is a natural question: *Is the complex contact form globally defined?* Kobayashi proved that the complex contact form is globally defined if and only if first Chern class of the manifold is zero. If complex contact form is globally defined then the manifold is called strict complex contact manifold. In 1980s there was an important development on Riemannian geometry of complex contact manifolds. Ishihara and Konishi [3] introduced complex almost contact metric structure and gave normality. We recall this normality by IK-normality. They also proved that under normality condition the base manifold is Kähler. In 2000 Korkmaz extended this definition.

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If the complex almost contact metric manifold is normal by Korkmaz's sense it is called *a normal complex contact metric manifold*, and in this study we use this notion. Also Korkmaz [4, 5, 6] gave curvature properties, $\mathcal{GH}-$sectional curvature, complex contact space form, complex (κ, μ) spaces and nullity conditions.
In recent years, there are some works on normal complex contact metric manifolds which are related to Korkmaz's normality. Blair and one of the present authors studied energy and corrected energy of vertical distribution for normal complex contact metric manifolds in [8, 7]. Fetcu studied an adapted connection on a strict complex contact manifolds and harmonic maps between complex Sasakian manifolds in [9, 10]. Blair and Molina [11] studied conformal and Bochner curvature tensor of normal complex contact metric manifolds. In 2012, Blair and Mihai studied on complex $(\kappa, \mu)-$space and they studied on locally symmetric condition of normal complex contact metric manifolds [12, 13]. Present authors [14] gave new results on curvature properties and normality conditions. Quasi-conformal, concircular and conharmonic flatness of normal complex contact metric manifolds are studied by presents authors [15] and they proved there are no normal complex contact metric manifolds under these tensors flatness.
The submanifold theory of complex contact manifolds has not studied yet, effectively. This is an area of awaiting attention, with many open problems. Turgut Vanlı studied on this subject [16, 17]. With above reasons our aim, is to give an introduction for the special submanifolds of complex contact manifolds. We take into consider the normality notion is given by Korkmaz [5]. By this way, our paper is organized as follow. The first section is on fundamental facts on complex contact manifolds. In the second section we give the definition for a semi-invariant submanifold of a normal complex contact metric manifold and obtain some relations. Finally, we give the integrability conditions of distributions in the last section .

2 Preliminaries

In this section we give a survey for complex contact manifolds. For further information we refer to reader [18] and [5].

Let (M, ω_i) be a complex contact manifold. For every $p \in M$ we have a subspace of T_pM by kernel of ω_i:

$$\mathcal{H}_i = \{K_p : \omega_i(K_p) = 0, K_p \in T_pM\}.$$

Then on $\mathcal{U}_i \cap \mathcal{U}_j \neq 0$ we have $\mathcal{H}_i = \mathcal{H}_j$ and so $\mathcal{H} = \cup \mathcal{H}_i$. $\mathcal{H}$ is well-defined, $2m-$complex dimensional non-integrable subbundle on M and it is called the contact subbundle or the horizontal subbundle. Let (M, ω_i) be a complex contact manifold. For every $p \in M$ we have a subspace of T_pM by kernel of ω_i:

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Ishihara and Konishi [3] proved that M admits always an almost contact structure of C^∞. They also give the Hermitian metric. An odd complex $2m+1-$dimensional complex manifold with Hermitian metric and almost contact structure is called *complex almost contact metric manifold.*

Let M be a odd complex $2m + 1-$dimensional complex manifold with complex structure J, Hermitian metric g, and $\mathcal{A} = \{\mathcal{U}_i\}$ be an open covering of M with coordinate neighborhoods $\{\mathcal{U}_i\}$. If M satisfies the following two conditions then it is called a *complex almost contact metric manifold*:

1. In each $\mathcal{U}_i$ there exist 1-forms u_i and $v_i = u_i \circ J$, with dual vector fields U_i and $V_i = -JU_i$ and $(1,1)$ tensor fields G_i and $H_i = G_iJ$ such that

$$H_i^2 = G_i^2 = -I + u_i \otimes U_i + v_i \otimes V_i \tag{2.1}$$

$$G_iJ = -JG_i, \quad GU_i = 0,$$

$$g(K, G_iL) = -g(G_iK, L).$$

2. On $\mathcal{U}_i \cap \mathcal{U}_j \neq \varnothing$ we have

$$\begin{aligned} u_j &= cu_i - dv_i, \quad v_j = du_i + cv_i, \\ G_j &= cG_i - dH_i, \quad H_j = dG_i + cH_i \end{aligned}$$

where c and d are functions on $\mathcal{U}_i \cap \mathcal{U}_j$ with $c^2 + d^2 = 1$ [3].

By direct computation we have

$$\begin{aligned} H_iG_i &= -G_iH_i = J_i + u_i \otimes V_i - v_i \otimes U_i \\ J_iH_i &= -H_iJ_i = G_i \\ G_iU_i &= G_iV_i = H_iU_i = H_iV_i = 0 \\ u_iG_i &= v_iG_i = u_iH_i = v_iH_i = 0 \\ J_iV_i &= U_i, \; g(U_i, V_i) = 0 \\ g(H_iX, Y) &= -g(X, H_iY). \end{aligned}$$

Since u_i and v_i are dual to the vector fields U_i and V_i , on $\mathcal{U}_i \cap \mathcal{U}_j$ we have $U_j = aU_i - bV_i$ and $V_j = bU_i + aV_i$. Also since $a^2 + b^2 = 1$, $U_j \wedge V_j = U_i \wedge V_i$. Thus U and V determine a global vertical distribution $\mathcal{V}$ by $\xi_i = U_i \wedge V_i$ which is typically assumed to be integrable. Moreover $\mathcal{V}$ is complex line bundle $TM/\mathcal{H}$. Then we have $TM = \mathcal{H} \oplus \mathcal{V}$.
From now on we will not use subscript for shortness, if $\mathcal{U}_i$ is understood.

In addition, we have

$$\begin{aligned} du(K, L) = g(K, GL) + (\sigma \wedge v)(K, L), \\ dv(K, L) = g(K, HL) - (\sigma \wedge u)(K, L) \end{aligned}$$

where $\sigma(K) = g(\nabla_K U, V)$, and ∇ being the Levi-Civita connection of g [3].

Ishihara and Konishi [3] defined following tensors ;

$$\begin{aligned} S(K, L) &= [G, G](K, L) + 2g(K, GL)U - 2g(K, HL)V \\ &\quad +2(v(L)HK - v(K)HL) + \sigma(GL)HK \\ &\quad -\sigma(GK)HL + \sigma(K)GHL - \sigma(L)GHK, \\ T(K, L) &= [H, H](K, L) - 2g(K, GL)U + 2g(K, HL)V \\ &\quad +2(u(L)GK - u(K)GL) + \sigma(HK)GL \\ &\quad -\sigma(HL)GK + \sigma(K)GHL - \sigma(L)GHK \end{aligned}$$

where

$$[G,G](K,L) = (\nabla_{GK}G)L - (\nabla_{GL}G)K - G(\nabla_K G)L + G(\nabla_L G)K$$

is the Nijenhuis torsion of G. Then they called an associated metric g normal if $S = T = 0$. It is called IK-normality. In generally we consider whether or not the complex analogue of the real normal contact examples are IK-normal. The canonical example, complex Heisenberg group is not IK-normal [18]. Because it is not Kähler. In 2000 Korkmaz [5] gave a weaker definition. In this sense, M is normal if the following two conditions are satisfied [5] :

1. $S(K,L) = T(K,L) = 0$ for all K, L in $\mathcal{H}$,
2. $S(K,U) = T(K,V) = 0$ for all K.

A normal complex contact metric manifold is semi-Kähler and the complex Heisenberg group is normal. In this paper, we use this notion of normality.

Korkmaz [5] obtained following equalities:

$$\nabla_K U = -GK + \sigma(K)V, \tag{2.2}$$

$$\nabla_K V = -HK - \sigma(K)U, \tag{2.3}$$

$$\nabla_U U = \sigma(U)V, \quad \nabla_U V = -\sigma(U)U \tag{2.4}$$

$$\nabla_V U = \sigma(V)V, \quad \nabla_V V = -\sigma(V)U,$$

$$\begin{aligned} d\sigma(GK,GL) &= d\sigma(HK,HL) \\ &= d\sigma(L,K) - 2u \wedge v(L,K)d\sigma(U,V). \end{aligned} \tag{2.5}$$

Theorem 2.1. [5] M is normal if and only if

$$\begin{aligned} g((\nabla_K G)L, Z) &= \sigma(K)g(HL,Z) + v(K)d\sigma(GZ,GL) \\ &\quad -2v(K)g(HGL,Z) - u(L)g(K,Z) \\ &\quad -v(L)g(JK,Z) + u(Z)g(K,L) \\ &\quad +v(Z)g(JK,L), \end{aligned} \tag{2.6}$$

$$\begin{aligned} g((\nabla_K H)L, Z) &= -\sigma(K)g(GL,Z) - u(K)d\sigma(HZ,HL) \\ &\quad -2u(K)g(HGL,Z) + u(L)g(JK,Z) \\ &\quad -v(L)g(K,Z) - u(Z)g(JK,L) \\ &\quad +v(Z)g(K,L). \end{aligned} \tag{2.7}$$

Also from above proposition we have

$$\begin{aligned} g((\nabla_K J)L, Z) &= u(K)(d\sigma(Z,GL) - 2g(HL,Z)) \\ &\quad + v(K)(d\sigma(Z,HL) + 2g(GL,Z)). \end{aligned} \tag{2.8}$$

Ishihara and Konishi [3] proved a normality condition by the term of he covariant derivatives of G and H. In [14] we obtain following theorem for a normal complex contact metric manifold .

Theorem 2.2. M is normal if and only if the covariant derivative of G and H have the following forms:

$$\begin{aligned}(\nabla_K G)L &= \sigma(K)HL - 2v(K)JL - u(L)K \\ &\quad -v(L)JK + v(K)\left(2JL_0 - (\nabla_U J)GL_0\right) \\ &\quad +g(K,L)U + g(JK,L)V \\ &\quad -d\sigma(U,V)v(K)\left(u(L)V - v(L)U\right), \end{aligned} \tag{2.9}$$

$$\begin{aligned}(\nabla_K H)L &= -\sigma(K)GL + 2u(K)JL + u(L)JK \\ &\quad -v(L)X + u(K)\left(-2JL_0 - (\nabla_U J)GL_0\right) \\ &\quad -g(JK,L)U + g(K,L)V \\ &\quad +d\sigma(U,V)u(K)\left(u(L)V - v(L)U\right) \end{aligned} \tag{2.10}$$

where $K = K_0 + u(K)U + v(K)V$ and $L = L_0 + u(L)U + v(L)V, K_0, L_0 \in \mathcal{H}$.

From this theorem we have

$$\begin{aligned}(\nabla_K J)L = &-2u(K)HL + 2v(K)GL + u(K)\left(2HL_0 + (\nabla_U J)L_0\right) \\ &+ v(K)\left(-2GL_0 + (\nabla_U J)JL_0\right).\end{aligned}$$

3 Fundamental facts on submanifolds of normal complex contact metric manifolds

Let $(\bar{M}^{4m+2}, \bar{G}, \bar{H}, \bar{J}, \bar{U}, \bar{V}, \bar{u}, \bar{v}, \bar{g})$ be a normal complex contact metric manifold, M be a $(n+2)-$real dimensional complex submanifold of $\bar{M}$ and $\bar{U}, \bar{V}$ be tangent to M, where n must be even. The Gauss formula is given by

$$\bar{\nabla}_K L = \nabla_K L + \mathbf{h}(K,L). \tag{3.1}$$

$\mathbf{h}$ is called the second fundamental form, and it is defined by:

$$\mathbf{h}(K,L) = \sum_{\alpha=1}^{r}(h^\alpha(K,L)N_\alpha + k^\alpha(K,L)\bar{J}N_\alpha).$$

where $r = \frac{4m-n}{2}$. We have the Wiengarten formulas which are given by

$$\bar{\nabla}_K N = -A_N K + \nabla^\perp_K N \tag{3.2}$$

$$\bar{\nabla}_K \bar{J}N = -A_{\bar{J}N} K + \nabla^\perp_K \bar{J}N \tag{3.3}$$

where A_N and $A_{\bar{J}N}$ are fundamental forms related to N and $\bar{J}N$. Also for $s^\alpha(K), t^\alpha(K)$ and $\tilde{s}^\alpha(K), \tilde{t}^\alpha(K)$ coefficients

$$\nabla^\perp_K N = \sum_{\alpha=1}^{r}\left(s^\alpha(K)N_\alpha + t^\alpha(K)\bar{J}N_\alpha\right) \text{ and } \nabla^\perp_K \bar{J}N = \sum_{\alpha=1}^{r}\left(\tilde{s}^\alpha(K)N_\alpha + \tilde{t}^\alpha(K)\bar{J}N_\alpha\right)$$

$\bar{\nabla}, \nabla$ and $\nabla^\perp$ are the Riemannian, induced connection and induced normal connections on $\bar{M}$, M and the normal bundle $TM^\perp$ of M, respectively. By easy computation we get following result.

Corollary 3.1. For any $K, L \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$ we have $\bar{g}(\mathbf{h}(K,L), N) = \bar{g}(A_N K, L)$.

The mean curvature μ of M is defined by $\mu = \frac{trace\ \mathbf{h}}{dim\ M}$. M is a totally umbilical submanifold if

$$\mathbf{h}(K,L) = g(K,L)\mu \tag{3.4}$$

for all $K, L \in \Gamma(TM)$.
Since $\bar{U}, \bar{V} \in \Gamma(TM)$ we can write $TM = sp\{\bar{U}, \bar{V}\} \oplus sp\{\bar{U}, \bar{V}\}^{\perp}$ where $sp\{\bar{U}, \bar{V}\}$ is the distribution spanned by $\bar{U}, \bar{V}$ and $sp\{\bar{U}, \bar{V}\}^{\perp}$ is the complementary orthogonal distribution of $sp\{\bar{U}, \bar{V}\}$ in M. Then for any vector field K is tangent to M we have $\bar{G}K \in sp\{\bar{U}, \bar{V}\}^{\perp}$ and $\bar{H}K \in sp\{\bar{U}, \bar{V}\}^{\perp}$. Let define following projections;

$$P : \Gamma(TM) \to \Gamma(TM) \quad , \quad Q : \Gamma(TM) \to \Gamma(TM)^{\perp}.$$

Then we can write

$$\bar{G}K = PK + QK \tag{3.5}$$

where PK and QK are the tangential and normal part of $\bar{G}K$, respectively. Since $\bar{H} = \bar{G}\bar{J}$ we have

$$\bar{H}K = P\bar{J}K + Q\bar{J}K. \tag{3.6}$$

Defined in this way P is an isomorphism on $\Gamma(TM)$ and Q is a normal valued 1-form on $\Gamma(TM)$. Therefore one can define two distributions for $p \in M$ as follows

$$\mathcal{D}_p = ker\{Q|_{sp\{\bar{U},\bar{V}\}^{\perp}}\} = \{K_p \in sp\{\bar{U}, \bar{V}\}^{\perp} : Q(K_p) = 0\}$$
$$\mathcal{D}_p^{\perp} = ker\{P|_{sp\{\bar{U},\bar{V}\}^{\perp}}\} = \{K_p \in sp\{\bar{U}, \bar{V}\}^{\perp} : P(K_p) = 0\}.$$

The following result is directly obtained from the definition of $\mathcal{D}_p$ and $\mathcal{D}_p^{\perp}$.

Proposition 3.2. $\mathcal{D}_p$ ve $\mathcal{D}_p^{\perp}$ are orthogonal subspaces of T_pM.

On the other hand for any vector field N normal to M we put

$$\bar{G}N = BN + CN \tag{3.7}$$

and

$$\bar{H}N = B\bar{J}N + C\bar{J}N. \tag{3.8}$$

where BN, $B\bar{J}N$ are tangential parts and CN, $C\bar{J}N$ are normal parts of $\bar{G}N$, $\bar{H}N$, respectively. Therefore we have projections

$$B : \Gamma(TM^{\perp}) \to \Gamma(TM)\ and\ C : \Gamma(TM^{\perp}) \to \Gamma(TM^{\perp}).$$

4 Semi-invariant submanifolds of normal complex contact metric manifolds

CR-submanifolds are important classes of complex submanifold theory. Similar to the definition of CR-submanifold, a semi-invariant submanifold of a Sasakian manifold was defined by Bejancu and Papaghuic [19]. We give an analogue definition for complex contact case.

Let $(\bar{M}^{4m+2}, \bar{G}, \bar{H}, \bar{J}, \bar{U}, \bar{V}, \bar{u}, \bar{v}, \bar{g})$ be a normal complex contact metric manifold, and M be a complex submanifold of $\bar{M}$. If the dimensions of $\mathcal{D}_p$ and $\mathcal{D}_p^{\perp}$ are constant along to M and

$$\mathcal{D} : p \to \mathcal{D}_p \quad , \quad \mathcal{D}^{\perp} : p \to \mathcal{D}_p^{\perp}$$

are differentiable then M is called a semi-invariant submanifold of $\bar{M}$. Bejancu and Papaghuic proved two results (Proposition 1.1 and Proposition 1.2 in [19]) about invariance of these distributions. Similarly we obtain following propositions for complex contact case.

Proposition 4.1. The distribution $\mathcal{D}$ is the maximal invariant distribution in $sp\{\bar{U}, \bar{V}\}^{\perp}$; that is, we have

1. $\bar{G}\mathcal{D}_p = \bar{H}\mathcal{D}_p = \mathcal{D}_p$
2. If $\mathcal{D}'_p \subset sp\{\bar{U}, \bar{V}\}^{\perp}$ and $\bar{G}\mathcal{D}'_p = \bar{H}\mathcal{D}'_p = \mathcal{D}'_p$ then we have $\mathcal{D}'_p \subset \mathcal{D}_p$.

Proposition 4.2. The distribution $\mathcal{D}_p^{\perp}$ is the maximal anti-invariant distribution in $sp\{\bar{U}, \bar{V}\}^{\perp}$; that is, we have

1. $\bar{G}\mathcal{D}_p^{\perp} \subset T_p^{\perp}M$, $\bar{H}\mathcal{D}_p^{\perp} \subset T_p^{\perp}M$
2. If $\mathcal{D}''_p \subset sp\{\bar{U}, \bar{V}\}^{\perp}$ and $\bar{G}\mathcal{D}''_p \subset T_p^{\perp}M$, $\bar{H}\mathcal{D}''_p \subset T_p^{\perp}M$ then we have $\mathcal{D}''_p \subset \mathcal{D}_p^{\perp}$ for any $p \in M$.

In real Sasakian geometry, Bejancu and Papaghuic [20] gave an equivalent definition by using invariance of $\mathcal{D}_p$, $\mathcal{D}_p^{\perp}$. Similarly by considering the Proposition 4.1 and Proposition 4.2 we get an equivalent definition.

Definition 4.3. Let $(\bar{M}^{4m+2}, \bar{G}, \bar{H}, \bar{U}, \bar{V}, \bar{u}, \bar{v}, g)$ be a normal complex contact metric manifold, M^{n+2} be a complex submanifold of $\bar{M}$ and $\bar{U}, \bar{V}$ be tangent to M. If following conditions are satisfied then M is called a semi-invariant submanifold.

1. $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus sp\{\bar{U}, \bar{V}\}$.
2. The distribution $\mathcal{D}$ is invariant by $\bar{G}$ and $\bar{H}$; that is, $\bar{G}\mathcal{D} = \mathcal{D}$ and $\bar{H}\mathcal{D} = \mathcal{D}$.
3. The distribution $\mathcal{D}^{\perp}$ is anti-invariant by $\bar{G}$ and $\bar{H}$; that is, $\bar{G}\mathcal{D}^{\perp} \subset TM^{\perp}$ and $\bar{H}\mathcal{D}^{\perp} \subset TM^{\perp}$.

Since $\bar{G}\bar{H}K = \bar{J}K$, for any vector field K in $\Gamma(\mathcal{D})$ or $\Gamma(\mathcal{D}^{\perp})$ above conditions are also satisfied for $\bar{J}$.

Let M be a semi-invariant submanifold of a normal complex contact metric manifold $\bar{M}$. If $dim\mathcal{D} = 0$ then M is called an anti-invariant submanifold of $\bar{M}$, and if $dim\mathcal{D}^{\perp} = 0$ then M is called an invariant submanifold of $\bar{M}$. If $\bar{G}\mathcal{D}^{\perp} = \bar{H}\mathcal{D}^{\perp} = TM^{\perp}$, then M is called generic submanifold of $\bar{M}$ [21].

For a semi-invariant submanifold M of a normal complex contact metric manifold $\bar{M}$, the projection morphisms of TM to $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are denoted by φ and ψ, respectively . Then for all $K \in \Gamma(TM)$ we can write

$$K = \varphi K + \psi K + \bar{u}(K)\bar{U} + \bar{v}(K)\bar{V} \tag{4.1}$$

where φK and ψK are tangential and normal parts of K, respectively. Also we have

$$\bar{J}K = \varphi\bar{J}K + \psi\bar{J}K + \bar{v}(K)\bar{U} - \bar{u}(K)\bar{V}. \tag{4.2}$$

Similarly for $N, \bar{J}N \in TM^{\perp}$ we have

$$N = tN + fN \text{ and } \bar{J}N = t\bar{J}N + f\bar{J}N$$

where $tN, t\bar{J}N$ is tangential part, and $fN, f\bar{J}N$ is the normal part of $N, \bar{J}N$. On the other hand from (3.7) and (3.8) we have $BN \in \Gamma(\mathcal{D}^{\perp}), B\bar{J}N \in \Gamma(\mathcal{D}^{\perp})$, $CN \in \Gamma(TM^{\perp})$ and $C\bar{J}N \in \Gamma(TM^{\perp})$. Thus we obtain an $f-$structure on the normal bundle by following same steps with the proof of Proposition 1.3 in [20].

From now on we will denote a semi-invariant submanifold of a normal complex contact metric manifold by M.

Proposition 4.4. On the normal bundle of M there exist an $f-$structure C.

For M we have following decomposition of normal space $TM^{\perp}$:

$$TM^{\perp} = \bar{G}\mathcal{D}^{\perp} \oplus \bar{H}\mathcal{D}^{\perp} \oplus \bar{J}\mathcal{D}^{\perp} \oplus \vartheta.$$

We can take an orthonormal frame

$$\{e_1, e_2, ..., e_m, \bar{G}e_1, \bar{G}e_2, ..., \bar{G}e_m, \bar{H}e_1, \bar{H}e_2, ..., \bar{H}e_m, \bar{J}e_1, \bar{J}e_2, ..., \bar{J}e_m, \bar{U}, \bar{V}\}$$

of $\bar{M}$ such that $\{e_1, e_2, ..., e_n\}$ are tangent to M. Therefore the set $\{e_1, e_2, ..., e_n, e_{n+1} = \bar{U}, e_{n+2} = \bar{V}\}$ is an orthonormal frame of M. We can consider $\{e_1, e_2, ..., e_n\}$ such that $\{e_1, e_2, ..., e_p\}$ is an orthonormal frame of $\mathcal{D}^{\perp}$, $\{e_{p+1}, e_{p+2}, ..., e_n\}$ is an orthonormal frame of $\mathcal{D}$. Moreover we can take $\{e_{n+3}, ..., e_{4m-n}\}$ as an orthonormal frame of $TM^{\perp}$ such that $\{e_{n+3}, ..., e_{n+2+3p}\}$ is an orthonormal frame of $\bar{G}\mathcal{D}^{\perp} \oplus \bar{H}\mathcal{D}^{\perp} \oplus \bar{J}\mathcal{D}^{\perp}$ and $\{e_{n+3+3p}, e_{n+4+3p}, ..., e_{4m+2}\}$ is an orthonormal frame of ϑ. From the definition of semi-invariant manifold we can take $e_{n+3} = \bar{G}e_1$, $e_{n+4} = \bar{G}e_2$,..., $e_{n+2+p} = \bar{G}e_p$, $e_{n+3+p} = \bar{H}e_1$, $e_{n+4+p} = \bar{H}e_2$,..., $e_{n+2+2p} = \bar{H}e_p$, $e_{n+3+2p} = \bar{J}e_1$, $e_{n+4+2p} = \bar{J}e_2$,..., $e_{n+2+3p} = \bar{J}e_p$. Therefore we have following orthonormal basis:

$$\begin{aligned} \mathcal{D} &= sp\{e_{\frac{p+1}{4}}, e_{\frac{p+2}{4}}, ..., e_{\frac{n-3}{4}}, \bar{G}e_{\frac{p+1}{4}}, \bar{G}e_{\frac{p+5}{4}}, ..., \bar{G}e_{\frac{n-3}{4}}, \bar{H}e_{\frac{p+1}{4}}, \\ &\quad \bar{H}e_{\frac{p+2}{4}}, ..., \bar{H}e_{\frac{n-3}{4}}, \bar{J}e_{\frac{p+1}{4}}, \bar{J}e_{\frac{p+2}{4}}, ..., \bar{J}e_{\frac{n-3}{4}}\} \\ \mathcal{D}^{\perp} &= sp\{e_1, e_2, ..., e_p\} \end{aligned}$$

and

$$\begin{aligned} \bar{G}\mathcal{D}^{\perp} \oplus \bar{H}\mathcal{D}^{\perp} \oplus \bar{J}\mathcal{D}^{\perp} &= sp\{\bar{G}e_1, \bar{G}e_2, ..., \bar{G}e_p, \bar{H}e_1, \bar{H}e_2, ..., \bar{H}e_p, \\ &\quad \bar{J}e_1, \bar{J}e_2, ..., \bar{J}e_p\} \\ \vartheta &= sp\{e_{p+1}, e_{p+2}, ..., e_{\frac{4m-n+3p}{4}}, \bar{G}e_{p+1}, \\ &\quad \bar{G}e_{p+2}, ..., \bar{G}e_{\frac{4m-n+3p}{4}}, \bar{H}e_{p+1}, \bar{H}e_{p+2}, \\ &\quad ..., \bar{H}e_{\frac{4m-n+3p}{4}}, \bar{J}e_{p+1}, \bar{J}e_{p+2}, ..., \bar{J}e_{\frac{4m-n+3p}{4}}\}. \end{aligned}$$

For M we compute covariant derivatives of $\bar{G}, \bar{H}, \bar{J}$ by given tangential and normal components. From (2.6), (2.7,) (2.8), (3.1), (3.2), (3.3), (3.5), (3.6), (3.7) and (3.8) and by easy computation we have following lemmas.

Lemma 4.5. For any $K, L \in \Gamma(TM)$ we have

$$\begin{aligned}
\varphi\nabla_K PL - \varphi A_{QL}K &= P\nabla_K L - \bar{u}(L)\varphi K + \bar{\sigma}(K)P\bar{J}L \\
&\quad - 2\bar{v}(K)\varphi\bar{J}L - \bar{v}(L)\varphi\bar{J}K + 2\bar{v}(K)\varphi\bar{J}L_0 \\
&\quad - \bar{v}(K)(\varphi\nabla_{\bar{U}}\bar{J}PL_0 - \bar{J}\varphi\nabla_{\bar{U}}PL_0 \\
&\quad - \varphi A_{\bar{J}QL_0}\bar{U} + \bar{J}\varphi A_{QL_0}\bar{U}),
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
\psi\nabla_K PL - \psi_{QL}K &= B\mathbf{h}(K,L) + \bar{\sigma}(K)Q\bar{J}L \\
&\quad - 2\bar{v}(K)\psi\bar{J}L - \bar{u}(L)\psi K - \bar{v}(L)\psi\bar{J}K \\
&\quad + \bar{v}(K)\psi\bar{J}L_0 - \bar{v}(K)(\psi\nabla_{\bar{U}}\bar{J}PL_0 - \bar{J}\psi\nabla_{\bar{U}}PL_0 \\
&\quad - \psi A_{\bar{J}QL_0}\bar{U} + \bar{J}\psi A_{QL_0}\bar{U} - B\bar{J}C\mathbf{h}(\bar{U}, PL_0)),
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
\bar{u}(\nabla_K PL - A_{QL}K) &= \bar{g}(\varphi K, \varphi L) + \bar{g}(\psi K, \psi L) \\
&\quad + (d\bar{\sigma}(\bar{U}, \bar{V}) - 2)\bar{v}(K)\bar{v}(L) - \bar{v}(K)(\bar{u}(\nabla_{\bar{U}}\bar{J}PL_0 \\
&\quad - A_{\bar{J}QL_0}\bar{U}) + \bar{v}(A_{QL_0}\bar{U} - \nabla_{\bar{U}}PL_0)),
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
\bar{v}(\nabla_K PL - A_{QL}K) &= \bar{g}(\varphi\bar{J}K, \varphi L) + \bar{g}(\psi\bar{J}K, \psi L) \\
&\quad - (d\bar{\sigma}(\bar{U}, \bar{V}) - 2)\bar{v}(K)\bar{u}(L) - \bar{v}(K)(\bar{v}(\nabla_{\bar{U}}\bar{J}PL_0 \\
&\quad - A_{\bar{J}QL_0}\bar{U}) + \bar{u}(\nabla_{\bar{U}}PL_0 + A_{QL_0}\bar{U}),
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
\mathbf{h}(K, PL) - C\mathbf{h}(K,L) + Q\nabla_K L &= \nabla^{\perp}_K QL - \bar{v}(K)(\mathbf{h}(\bar{U}, \bar{J}PL_0) \\
&\quad - Q\bar{J}B\mathbf{h}(\bar{U}, PL_0) - C\bar{J}C\mathbf{h}(\bar{U}, PL_0) \\
&\quad + \nabla^{\perp}_{\bar{U}}\bar{J}QL_0 - \bar{J}\nabla^{\perp QL_0}_{\bar{U}}).
\end{aligned} \tag{4.7}$$

Lemma 4.6. For arbitrary vector fields K and L on M we have

$$\begin{aligned}
\varphi\nabla_K P\bar{J}L - \varphi A_{Q\bar{J}L}K &= P\bar{J}\nabla_K L - \bar{\sigma}(K)PL + 2\bar{u}(K)\varphi\bar{J}L \\
&\quad + \bar{u}(L)\varphi\bar{J}K - \bar{v}(L)\varphi K - 2\bar{u}(K)\varphi\bar{J}L_0 \\
&\quad - \bar{u}(K)(\varphi\nabla_{\bar{U}}\bar{J}PL_0 - \bar{J}\varphi\nabla_{\bar{U}}PL_0 \\
&\quad - \varphi A_{\bar{J}QL_0}\bar{U} + \bar{J}\varphi A_{QL_0}\bar{U}),
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
\psi\nabla_K P\bar{J}L - \psi A_{Q\bar{J}L}K &= B\bar{J}\mathbf{h}(K,L) - \sigma(K)QL + 2\bar{u}(K)\psi\bar{J}L \\
&\quad + \bar{u}(L)\psi\bar{J}K - \bar{v}(L)\psi K - 2\bar{u}(K)\psi\bar{J}L_0 \\
&\quad - \bar{u}(K)(\psi\nabla_{\bar{U}}\bar{J}PL_0 - \bar{J}\psi\nabla_{\bar{U}}PL_0 \\
&\quad - \psi A_{\bar{J}QL_0}\bar{U} + \bar{J}\psi A_{QL_0}\bar{U} - B\bar{J}C\mathbf{h}(\bar{U}, PL_0)),
\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
\bar{u}(\nabla_K P\bar{J}L - A_{Q\bar{J}L}K) &= -\bar{g}(\varphi\bar{J}K, \varphi L) - \bar{g}(\psi\bar{J}K, \psi L) \\
&- (d\sigma(\bar{U}, \bar{V}) - 2)\bar{v}(K)\bar{v}(L) + \bar{u}(K)(-\bar{u}(\nabla_{\bar{U}}\bar{J}PL_0 \\
&- A_{\bar{J}QL_0}\bar{U}) + \bar{v}(\nabla_{\bar{U}}PL_0 + A_{QL_0}\bar{U}),
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
\bar{v}(\nabla_K P\bar{J}L - A_{Q\bar{J}L}K) &= \bar{g}(\varphi K, \varphi L) + \bar{g}(\psi K, \psi L) \\
&+ (d\sigma(\bar{U}, \bar{V}) - 2)\bar{u}(K)\bar{u}(L) - \bar{u}(K)(\bar{u}(\nabla_{\bar{U}}PL_0 \\
&+ A_{QL_0}\bar{U}) + \bar{v}(\nabla_{\bar{U}}\bar{J}PL_0 - A_{\bar{J}QL_0}\bar{U}) \\
&+ \bar{v}(\nabla_{\bar{U}}PL_0) + \bar{u}(A_{\bar{J}QL_0}\bar{U}) + \bar{v}(A_{\bar{J}QL_0}\bar{U})),
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
\mathbf{h}(K, P\bar{J}L) - C\bar{J}\mathbf{h}(K, L) &= -Q\bar{J}\nabla_K L - \nabla^{\perp}_K Q\bar{J}L \\
&- \bar{u}(K)(\mathbf{h}(\bar{U}, \bar{J}PL_0) - Q\bar{J}B\mathbf{h}(\bar{U}, PL_0) \\
&- C\bar{J}C\mathbf{h}(\bar{U}, PL_0) + \nabla^{\perp}_{\bar{U}}\bar{J}QL_0 - \bar{J}\nabla^{\perp QL_0}_{\bar{U}}).
\end{aligned} \tag{4.12}
$$

Lemma 4.7. For any $K, L \in \Gamma(TM)$ we have

$$
\begin{aligned}
\varphi\nabla_K BN - \varphi A_{CN}K - PA_N K &= \bar{v}(K)(\varphi A_{\bar{J}BN}\bar{U} + \varphi\bar{J}\nabla_{\bar{U}}BN \\
&- \varphi A_{\bar{J}CN}\bar{U} - \varphi\bar{J}A_{CN}\bar{U}), \\
\psi\nabla_K BN - \psi A_{CN}K - B\nabla^{\perp}_K N &= \bar{\sigma}(K)B\bar{J}N + \bar{v}(K)(\psi A_{\bar{J}BN}\bar{U} \\
&+ \psi\bar{J}\nabla_{\bar{U}}BN + B\bar{J}C\mathbf{h}(\bar{U}, BN) + \psi A_{\bar{J}CN}\bar{U} \\
&- \psi\bar{J}A_{CN}\bar{U} + B\bar{J}C\nabla^{\perp}_{\bar{U}}CN),
\end{aligned}
$$

$$
\begin{aligned}
\bar{u}(\nabla_K BN) - \bar{u}(A_{CN}K) &= \bar{v}(K)[\bar{u}(A_{\bar{J}BN}\bar{U}) + \bar{v}(\nabla_U BN) \\
&+ \bar{u}(A_{C\bar{J}N}\bar{U}) + \bar{v}(A_{CN}\bar{U})], \\
\bar{v}(\nabla_K BN) - \bar{v}(A_{CN}K) &= \bar{v}(K)[\bar{v}(A_{\bar{J}BN}\bar{U}) - \bar{u}(\nabla_U BN) \\
&+ \bar{v}(A_{\bar{J}CN}\bar{U}) + \bar{u}(A_{CN}\bar{U})],
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{h}(K, BN) + \nabla^{\perp}_K CN - QA_N K &= C\nabla^{\perp}_K N + \bar{\sigma}(K)C\bar{J}N - \bar{v}(K)[\nabla^{\perp}_{\bar{U}}B\bar{J}N \\
&- Q\bar{J}B\mathbf{h}(\bar{U}, BN) + \nabla^{\perp}_{\bar{U}}\bar{J}CN \\
&- Q\bar{J}C\nabla^{\perp}_{\bar{U}}CN - C\bar{J}C\nabla^{\perp}_{\bar{U}}CN].
\end{aligned}
$$

Lemma 4.8. For any $K, L \in \Gamma(TM)$ we have

$$
\begin{aligned}
\varphi\nabla_K B\bar{J}N - \varphi A_{C\bar{J}N}K + PA_N K &= \bar{u}(K)(-\varphi A_{\bar{J}BN}\bar{U} \\
&+ \varphi\bar{J}\nabla_U BN - \varphi A_{\bar{J}CN}\bar{U} + \varphi\bar{J}A_{CN}\bar{U}), \\
\psi\nabla_K B\bar{J}N - \psi A_{C\bar{J}N}K - B\bar{J}\nabla^{\perp}_K N &= \bar{\sigma}(K)BN + \bar{u}(K)(-\psi A_{\bar{J}BN}\bar{U} \\
&- \psi\bar{J}\nabla_{\bar{U}}BN - B\bar{J}C\mathbf{h}(\bar{U}, BN) - \psi A_{\bar{J}CN}\bar{U} \\
&+ \psi\bar{J}A_{CN}\bar{U} - B\bar{J}C\nabla^{\perp}_{\bar{U}}CN),
\end{aligned}
$$

$$
\begin{aligned}
\bar{u}(\nabla_K B\bar{J}N) - \bar{u}(A_{C\bar{J}N}K) &= -\bar{u}(K)[\bar{u}(A_{\bar{J}BN}\bar{U}) \\
&\quad + \bar{v}(\nabla_U BN) + \bar{u}(A_{C\bar{J}N}\bar{U}) + \bar{v}(A_{CN}\bar{U})], \\
\bar{v}(\nabla_K B\bar{J}N) - \bar{v}(A_{C\bar{J}N}K) &= -\bar{u}(K)[\bar{v}(A_{\bar{J}BN}\bar{U}) \\
&\quad - \bar{u}(\nabla_U BN) + \bar{v}(A_{\bar{J}CN}\bar{U}) - \bar{u}(A_{CN}\bar{U})], \\
\mathbf{h}(K, B\bar{J}N) + \nabla^\perp_K C\bar{J}N - QA_N K &= C\bar{J}\nabla^\perp_K N + \bar{\sigma}(K)CN + \bar{u}(K)[\nabla^\perp_{\bar{U}} B\bar{J}N \\
&\quad - Q\bar{J}B\mathbf{h}(\bar{U}, BN) - C\bar{J}C\mathbf{h}(\bar{U}, BN) \\
&\quad + \nabla^\perp_{\bar{U}}\bar{J}CN - Q\bar{J}C\nabla^\perp_{\bar{U}}CN - C\bar{J}C\nabla^\perp_{\bar{U}}CN].
\end{aligned}
$$

As we know the covariant derivatives of structure vector fields are important. In the following lemma we give the covariant derivatives of $\bar{U}$ and $\bar{V}$ with ∇ on M.

Lemma 4.9. For any $K, L \in \Gamma(TM)$ we have

$$
\begin{aligned}
\nabla_K \bar{U} &= -PK + \bar{\sigma}(K)\bar{V}, \quad \mathbf{h}(K, \bar{U}) = -QK \\
\nabla_K \bar{V} &= -P\bar{J}K - \bar{\sigma}(K)\bar{U}, \quad \mathbf{h}(K, \bar{V}) = -Q\bar{J}K.
\end{aligned}
$$

Proof. From (2.2) and (3.1) we get

$$
-\bar{G}K + \bar{\sigma}(K)\bar{V} = \nabla_K \bar{U} + \mathbf{h}(K, \bar{U}),
$$

and by consider tangent and normal components we obtain (4.13). Similarly from (2.3) and (3.1) we get (4.13). Q.E.D.

Also from these lemmas, we get following corollaries.

Corollary 4.10. For M we have

$$
\begin{aligned}
\mathbf{h}(K, \bar{U}) &= \mathbf{h}(K, \bar{V}) = 0 \\
\nabla_K \bar{U} &= -PK + \bar{\sigma}(K)\bar{V}, \;\; \nabla_K \bar{V} = -P\bar{J}K + \bar{\sigma}(K)\bar{U}
\end{aligned}
$$

for all $K \in \Gamma(\mathcal{D})$, and

$$
\begin{aligned}
\mathbf{h}(K, \bar{U}) &= -QK, \; \mathbf{h}(K, \bar{V}) = -Q\bar{J}K \\
\nabla_K \bar{U} &= \bar{\sigma}(K)\bar{V}, \;\; \nabla_K \bar{V} = -\bar{\sigma}(K)\bar{U}
\end{aligned}
$$

for all $K \in \Gamma(\mathcal{D}^\perp)$.

Corollary 4.11. For M we have

$$
\begin{aligned}
\mathbf{h}(\bar{U}, \bar{U}) &= \mathbf{h}(\bar{V}, \bar{V}) = \mathbf{h}(\bar{U}, \bar{V}) = 0 \\
\nabla_{\bar{U}} \bar{U} &= \bar{\sigma}(\bar{U})\bar{V}, \;\; \nabla_{\bar{V}} \bar{U} = \bar{\sigma}(\bar{V})\bar{V} \\
\nabla_{\bar{U}} \bar{V} &= -\bar{\sigma}(\bar{U})\bar{U}, \;\; \nabla_{\bar{V}} \bar{V} = -\bar{\sigma}(\bar{V})\bar{U}.
\end{aligned}
$$

5 Integrability of distributions

In the submanifold theory integrability of distributions is an important notion. In this work we have two distributions, $\mathcal{D}$ and $\mathcal{D}^{\perp}$. In this section we give some result about integrability of $\mathcal{D}$, $\mathcal{D}^{\perp}$, $\mathcal{D}\oplus\mathcal{D}^{\perp}$, $\mathcal{D}\oplus sp\{\bar{U},\bar{V}\}$ and $\mathcal{D}^{\perp}\oplus sp\{\bar{U},\bar{V}\}$. Although the horizontal distribution $\mathcal{H}$ is never involute, as we shall see some of above distributions are involute on M.

Lemma 5.1. For M we have

$$\bar{g}(A_{\bar{G}K}L,Z)=\bar{g}(A_{\bar{G}L}K,Z) \tag{5.1}$$
$$\bar{g}(A_{\bar{H}K}L,Z)=\bar{g}(A_{\bar{H}L}K,Z) \tag{5.2}$$
$$\bar{g}(A_{\bar{J}K}L,Z)=\bar{g}(A_{\bar{J}L}K,Z) \tag{5.3}$$

for all $K,L\in\Gamma(\mathcal{D})$, Z is tangent to M and $Z\notin sp\{\bar{U},\bar{V}\}$.

Proof. Let $K,L\in\Gamma(\mathcal{D})$ and $Z\in\Gamma(TM)$. Since $\bar{G}K=QK\in\Gamma(TM^{\perp})$, we have

$$\bar{\nabla}_L\bar{G}K=-A_{\bar{G}K}L+\nabla^{\perp}_L\bar{G}K \tag{5.4}$$

and

$$\bar{\nabla}_LZ=\nabla_LZ+\mathbf{h}(L,Z).$$

Thus we get

$$\bar{g}(\bar{\nabla}_LZ,\bar{G}K)=\bar{g}(\mathbf{h}(L,Z),\bar{G}K)$$

and since $\bar{G}K\in\Gamma(TM^{\perp})$ then $\bar{g}(\bar{\nabla}_LZ,\bar{G}K)+\bar{g}(Z,\bar{\nabla}_L\bar{G}K)=0$ and therefore from (5.4) we get

$$\bar{g}(A_{\bar{G}K}L,Z)=\bar{g}(\mathbf{h}(L,Z),\bar{G}K).$$

In addition since $\mathbf{h}$ is symmetric and from (5.4) we have

$$\begin{aligned}\bar{g}(A_{\bar{G}K}L,Z) &= -\bar{g}(\bar{G}\bar{\nabla}_ZL,K)\\ &= \bar{g}((\bar{\nabla}_Z\bar{G})L,K)-\bar{g}(\bar{\nabla}_Z\bar{G}L,K).\end{aligned}$$

From (2.6) and (2.5) we have

$$\bar{g}((\bar{\nabla}_Z\bar{G})L,K)=\bar{g}(d\sigma(L,K)\bar{V},Z)$$

and so we get

$$\bar{g}(A_{\bar{G}K}L,Z) = \bar{g}(d\sigma(L,K)\bar{V},Z)-\bar{g}(\bar{\nabla}_Z\bar{G}L,K).$$

On the other hand since $\bar{g}(\bar{G}L,K)=0$ and from (3.1) we have

$$\begin{aligned}\bar{g}(A_{\bar{G}K}L,Z) &=\bar{g}(d\sigma(L,K)\bar{V},Z)+\bar{g}(\bar{\nabla}_ZK,\bar{G}L)\\ &=\bar{g}(d\sigma(L,K)\bar{V},Z)+\bar{g}(\nabla_ZK+\mathbf{h}(Z,K),\bar{G}L)\\ &=\bar{g}(d\sigma(L,K)\bar{V},Z)+\bar{g}(\mathbf{h}(Z,K),\bar{G}L)\end{aligned}$$

and thus, from (5.5) we get

$$\bar{g}(A_{\bar{G}K}L, Z) = \bar{g}(d\sigma(L, K)\bar{V}, Z) + \bar{g}(A_{\bar{G}L}K, Z).$$

If $Z \notin sp\{\bar{U}, \bar{V}\}$ we get (5.1). By following same steps one can show (5.2), (5.3). Q.E.D.

Lemma 5.2. For all $K, L \in \Gamma(\mathcal{D}^{\perp})$ we have $[K, L] \in \Gamma(\mathcal{D} \oplus \mathcal{D}^{\perp})$.

Proof. Let $K, L \in \Gamma(D^{\perp})$. Then we have

$$\begin{aligned} \bar{g}([K, L], \bar{U}) &= \bar{g}(\bar{\nabla}_K L - \bar{\nabla}_L K, \bar{U}) \\ &= -\bar{g}(\bar{\nabla}_K \bar{U}, L) + \bar{g}(\bar{\nabla}_L \bar{U}, K). \end{aligned}$$

Therefore from (2.2) we have $\bar{g}([K, L], \bar{U}) = 0$. Also $\bar{g}([K, L], \bar{V}) = 0$ can be showed by similar way. So we obtain $[K, L] \in \Gamma(D \oplus D^{\perp})$.

Q.E.D.

Theorem 5.3. The anti-invariant distribution is involutive.

Proof. Let $K, L \in \Gamma(\mathcal{D}^{\perp})$. From (2.9) we have

$$(\bar{\nabla}_K \bar{G})L = \bar{\sigma}(K)\bar{H}L + \bar{g}(K, L)\bar{U}.$$

On the other hand $\bar{G}L \in \Gamma(D)$ and from (3.1) and (3.2) we have

$$-A_{\bar{G}L}K + \nabla^{\perp}_K L - \bar{G}\nabla_K L - \bar{G}\mathbf{h}(K, L) = \bar{\sigma}(K)\bar{H}L + \bar{g}(K, L)\bar{U}. \tag{5.5}$$

Substituting L by K in (5.5) and thus subtracting the obtained relations we get

$$-\bar{G}[K, L] = A_{\bar{G}L}K - A_{\bar{G}K}L - \nabla^{\perp}_K \bar{G}L - \nabla^{\perp}_L \bar{G}K + \bar{\sigma}(K)\bar{H}L - \bar{\sigma}(L)\bar{H}K.$$

Now we take an arbitrary normal section $N \in \Gamma(\vartheta)$ and, by using (2.6) and (3.2) we have

$$\bar{g}(\nabla^{\perp}_L \bar{G}K, N) = -\bar{g}(A_{\bar{G}N}L, K). \tag{5.6}$$

Substituting L by K in (5.6) and, subtracting the obtained relations , since $A_{\bar{G}N}$ is symmetric we have

$$\bar{g}(\nabla^{\perp}_K \bar{G}L - \nabla^{\perp}_L \bar{G}K, N) = 0.$$

Hence $\nabla^{\perp}_K \bar{G}L - \nabla^{\perp}_L \bar{G}K \in \bar{G}\mathcal{D}^{\perp} \oplus \bar{H}\mathcal{D}^{\perp} \oplus \bar{J}\mathcal{D}^{\perp}$. On the other hand for $Z \in \Gamma(\mathcal{D})$ from (5.6) we have

$$\bar{g}(-\bar{G}[K, L], \bar{G}Z) = 0$$

and therefore we get

$$\bar{g}([K, L], \bar{G}^2 Z) = \bar{g}([K, L], Z) = 0.$$

So we obtain $[K, L] \in \Gamma(\mathcal{D})$. Q.E.D.

Theorem 5.4. $\mathcal{D}^{\perp} \oplus sp\{\bar{U}, \bar{V}\}$ distribution is involute.

Proof. Let $K \in \Gamma(\mathcal{D}^{\perp})$ and $L \in \Gamma(\mathcal{D})$. Then from (2.2) we have

$$\bar{g}([K, \bar{U}], L) = -\bar{g}(\bar{\nabla}_{\bar{U}} K, L).$$

Now let take $Z \in \Gamma(\mathcal{D})$ such that $L = \bar{G}Z$ and by using (2.9) we have

$$(\bar{\nabla}_{\bar{U}} H)Z = \sigma(\bar{U})\bar{H}Z$$

and from (3.1) we get

$$\bar{g}([K, U], L) = \bar{g}(\bar{\nabla}_{\bar{U}} \bar{G}Z, K) = -\bar{g}(\nabla_{\bar{U}} Z, \bar{G}K) = 0.$$

Therefore $[K, \bar{U}] \in \mathcal{D}^{\perp} \oplus sp\{\bar{U}, \bar{V}\}$. Following by same steps one can show the $[K, \bar{U}] \in \mathcal{D}^{\perp} \oplus sp\{\bar{U}, \bar{V}\}$. Consequently by consider (5.3) the theorem is proved. Q.E.D.

Definition 5.5. If M is neither an invariant submanifold (i.e $dim\mathcal{D}^{\perp} = 0$) nor an anti-invariant submanifold (i.e $dim\mathcal{D} = 0$), then it is called a proper semi-invariant submanifold.

Theorem 5.6. The invariant distribution is never involute.

Proof. For $K, L \in \Gamma(\mathcal{D})$ from (2.2) we get

$$\bar{g}([K, L], \bar{U}) = 2\bar{g}(\bar{G}K, L)$$

and from (2.3) we have

$$\bar{g}([K, L], \bar{U}) = 2\bar{g}(\bar{G}K, L).$$

Let choose $L = \bar{H}K$ for all $L \in \Gamma(\mathcal{D})$ such that $\bar{H}K$ is a unit vector field. Thus the second fundamental form can not vanish. So $\mathcal{D}$ is not involute. Q.E.D.

From this theorem we have :

Corollary 5.7. Let M be a proper semi-invariant submanifold. Then the distribution $\mathcal{D} \oplus \mathcal{D}^{\perp}$ is never involute.

We need two following lemmas to get necessary and sufficient conditions for the integrability of $\mathcal{D} \oplus sp\{\bar{U}, \bar{V}\}$.

Lemma 5.8. Let M be a semi-invariant submanifold. Then, we have

$$\begin{aligned}
\bar{g}(\mathbf{h}(K, L), \bar{G}Z) &= \bar{g}(\nabla_K Z, \bar{G}L) \\
\bar{g}(\mathbf{h}(K, L), \bar{H}Z) &= \bar{g}(\nabla_K Z, \bar{H}L) \\
\bar{g}(\mathbf{h}(K, L), \bar{J}Z) &= \bar{g}(\nabla_K Z, \bar{J}L)
\end{aligned}$$

for all vector fields $K \in \Gamma(TM)$, $L \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D})$.

Proof. Let $N = \bar{G}Z$ then from (5.5) we have

$$\bar{g}(\mathbf{h}(K,L),\bar{G}Z) = \bar{g}(A_{\bar{G}Z}K,L) = -\bar{g}((\bar{\nabla}_K\bar{G})Z + \bar{G}\bar{\nabla}_K Z, L).$$

On the other hand from (2.6) we get

$$\bar{g}(\mathbf{h}(K,L),\bar{G}Z) = \bar{g}(\bar{\nabla}_K Z, \bar{G}L).$$

By following same steps, the equations: $\bar{g}(\mathbf{h}(K,L),\bar{H}Z) = \bar{g}(\nabla_K Z, \bar{H}L)$ and $\bar{g}(\mathbf{h}(K,L),\bar{J}Z) = \bar{g}(\nabla_K Z, \bar{J}L)$ can be obtained. Q.E.D.

Lemma 5.9. For M we have $[K,\bar{U}]$ and $[K,\bar{V}] \in \Gamma(\mathcal{D} \oplus sp\{\bar{U},\bar{V}\})$.

Proof. By using (3.1) and (4.13) we have

$$\bar{g}([K,\bar{U}],L) = \bar{g}(\nabla_{\bar{U}}L, K)$$

for each $L \in \Gamma(\mathcal{D}^{\perp})$ and $K \in \Gamma(\mathcal{D})$. Now we take $Z \in \Gamma(\mathcal{D})$ such that $K = \bar{G}Z$ and from (5.5) we get

$$\bar{g}(\nabla_{\bar{U}}L, K) = \bar{g}(\mathbf{h}(\bar{U},Z),\bar{G}L) = 0.$$

Thus $\bar{g}([K,\bar{U}],L) = 0$ and by following same steps we get $\bar{g}([K,\bar{V}],L) = 0$, it follows the assertion of the lemma. Q.E.D.

Theorem 5.10. The distribution $\mathcal{D} \oplus sp\{\bar{U},\bar{V}\}$ is involutive if and only if we have

$$\mathbf{h}(K,\bar{G}L) = \mathbf{h}(\bar{G}K,L). \tag{5.7}$$

Proof. From (4.7) we obtain

$$\mathbf{h}(K,PL) - C\mathbf{h}(K,L) + Q\nabla_K L = 0 \tag{5.8}$$

for all $K,L \in \Gamma(\mathcal{D})$. Since $\mathbf{h}$ is symmetric substituting L by K in (5.8) we get $\mathbf{h}(K,PL) - \mathbf{h}(L,PK) = Q[K,L]$. In this way $[K,L] \in \mathcal{D} \oplus sp\{\bar{U},\bar{V}\}$ if and only if (5.7) is satisfied. Taking into account (5.9), the proof is completed. Q.E.D.

Finally we obtain a result for total umbilical semi-invariant submanifold.

Theorem 5.11. If M is a total umbilical submanifold then M is an invariant submanifold.

Proof. Let M be a total umbilical semi-invariant submanifold. Then for $\forall Z \in \Gamma(\mathcal{D})^{\perp}$ from (3.4) we have

$$\mathbf{h}(Z,\bar{U}) = \bar{g}(Z,\bar{U})\mu = 0.$$

On the other hand from (4.13) we have $\mathbf{h}(Z,\bar{U}) - QZ = \bar{G}Z$. Thus $\bar{G}Z = 0$ and $\mathcal{D}^{\perp} = 0$. So M is an invariant submanifold. Q.E.D.

From above theorem we obtain following corollary.

Corollary 5.12. There does not exist total umbilical proper semi-invariant submanifold of a normal complex contact metric manifold.

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